

## Math Musings

### Bowling Balls and Binary Switches

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Back in the old programming days, my favorite line of code was if-then. If some condition is met, then do something specific - incredibly useful for making things happen exactly when you want them to happen. Well, recently I was using graphing software to generate curves and wanted to find an algebraic equivalent of an if-then command. Like a binary switch, it would return a value of 1 for a certain range of  $x$ -axis values, and return a value of zero for all other  $x$ -values. My desire was to be able to turn individual sub-equations on and off during the execution of a single graphing equation, while graphing a curve that has different functions at different places along the  $x$ -axis. Multiplying these different sub-equations by the binary switch equation would effectively turn them on and off wherever I wanted.

So I imagined a segment on the  $x$ -axis with a center at  $x=0$ , and a variable  $r$  defining an equal length along both the positive and negative axis of  $x$ . When the  $x$ -value is between  $-r$  and  $+r$ , I wanted an equation that would return a value of 1, and at all other  $x$  values would return a zero. An equation using absolute values does the trick - now I had a switch that could activate different equations at different places on the  $x$ -axis. Let's arbitrarily set the value of  $r$  to 0.5, giving the range between  $x=-0.5$  and  $x=+0.5$  that will be controlled separately by the binary switch. Let's call this binary switch  $\mathfrak{B}$ , so

$$\mathfrak{B} = \frac{(|x|-r) - (|x|+r)}{2(|x|-r)}$$

As you can see, when  $x \leq -r$  and  $x \geq +r$ , this binary switch formula will return a value of zero. When  $x > -r$  and  $x < +r$ , it will return a value of one. A use I had in mind was plotting a simple curve with the equation

$$-\frac{m}{(|t|+a)^2}$$
, which I will call *equation 1*. It gives the curve in fig. 1. I have chosen the arbitrary value of  $m=1.5$ ; and  $a=1$  to keep all values of the denominator  $\geq 1$ .

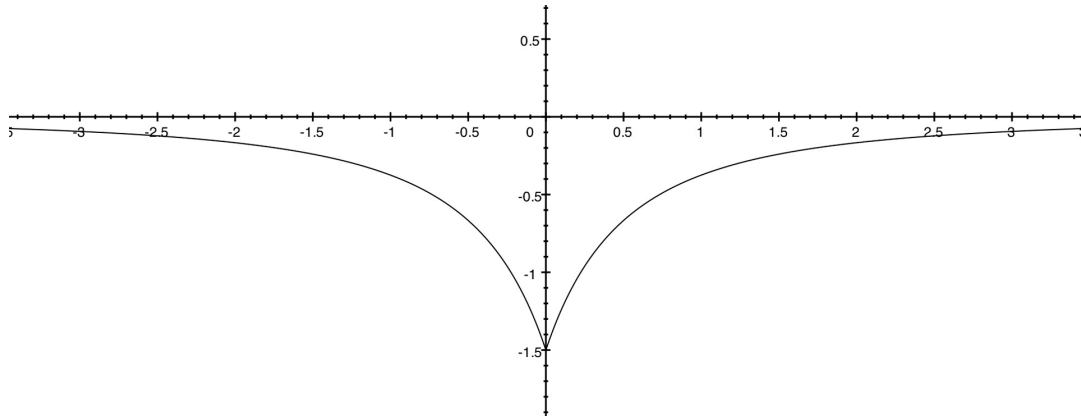


fig. 1

What I'd like to do is have this curve be plotted until the value of  $x=-r$  is reached, and then terminate, and then begin to plot again at  $x=+r$ , and continue. So I can use the binary switch to turn on and off equation 1. However, since the binary switch  $\mathfrak{I}$  returns a value of zero when  $x>-r$  and  $x<+r$ , that would turn off equation 1 when I need to turn it on. So I need to reverse the output of  $\mathfrak{I}$ , changing 1 to zero and zero to 1. This I can do by simply subtracting 1 from  $\mathfrak{I}$  and taking the absolute value, or  $|\mathfrak{I}-1|$ . That way, when  $\mathfrak{I}$  returns a zero,  $|\mathfrak{I}-1|$  returns 1, and when  $\mathfrak{I}$  returns a 1, then  $|\mathfrak{I}-1|$  returns zero. Now, since

$|\mathfrak{I}-1| = \left| \frac{(|x|-r) - ||x|-r||}{2(|x|-r)} - 1 \right|$ , I can use  $(|\mathfrak{I}-1|) \left( -\frac{m}{(|x|+a)^2} \right)$  to create the curve in figure 2 that successfully terminates at  $x=-r$  and starts again at  $x=+r$ .

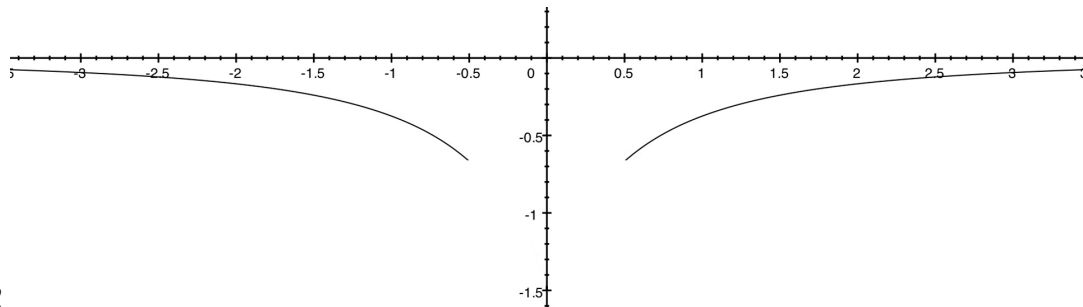


fig.2

Now for the second equation that will create a different curve in the missing spot along the x-axis, when  $x>-r$  and  $x<+r$ . I imagined a rubber sheet with a bowling ball on it, with its center at  $x=0$ . The curve down to the boundary of the ball has been created by equation 1, and the curve below the bowling ball will need a different equation, namely to create an arc of a circle cut by a chord at values  $x=-r$  and  $x=+r$ . We'll call the equation to make this curve *equation 2*. This curve, which follows the bottom of the bowling ball, can be created

by drawing a circle at a precise point using the following equation. The value of  $q$  defines the size of the circle, and the value of  $u$  determines how much of the arc of the circle will be drawn. To draw the entire circle,  $u$  must be  $\geq \pi$ .

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q \cos ut \\ q \sin ut + \left( -\frac{m}{(r+a)^2} + p \right) \end{bmatrix}, \text{ where } u \geq \pi \text{ and } t = -1 \dots 1$$

The trick in combining equation 1 and equation 2 is to find the center of the circle to be drawn, or point R in figure 3. The  $y$  coordinate for point P (and point Q) is returned by  $-m/(r+a)^2$  in equation 2 above, but we will need the value of  $p$  to find the circle's center. To find  $p$  I began by calculating the first derivative of the termination point of equation 1 at point P, or  $x=-r$ , to arrive at the instantaneous slope of the curve at point P.

So if we choose  $r=0.5$ , then using the power rule for equation 1, we have:  $2m(r+a)^{-3}$ ,  
 or  $\left(\frac{dy}{dx}\right)_{(x=-r)} = -0.888$ . The angle of the slope in relation to line  $r$  is  $\text{atan}\left(\frac{dy}{dx}\right)_{(x=-r)}$ ,  
 or  $-0.7266$ . Since line  $q$  is at right angles to the slope or tangent line, adding the absolute

values of  $\alpha_{sr}$  and  $\alpha_P$  (figure 3) must equal  $\pi/2$ . Because  $\text{atan}\left(\frac{dy}{dx}\right)_{(x=-r)}$  is negative,

we simply add  $\pi/2$ , so  $\frac{\pi}{2} + \text{atan}\left(\frac{dy}{dx}\right)_{(x=-r)} = \alpha_P$ , or  $\alpha_P = 0.8442$ . Now, knowing  $\alpha_Q$  is a right angle, we can find  $\alpha_R$ , which is  $\pi - (\alpha_P + \alpha_Q)$ , or  $0.7266$ .  $\alpha_R$  is also equivalent to  $\text{atan } dy/dx(x=-r)$ . Armed with this information, it is easy to use the trigonometric law of sines to get the  $x,y$  coordinates of point R, which is the center of the circle to be drawn.

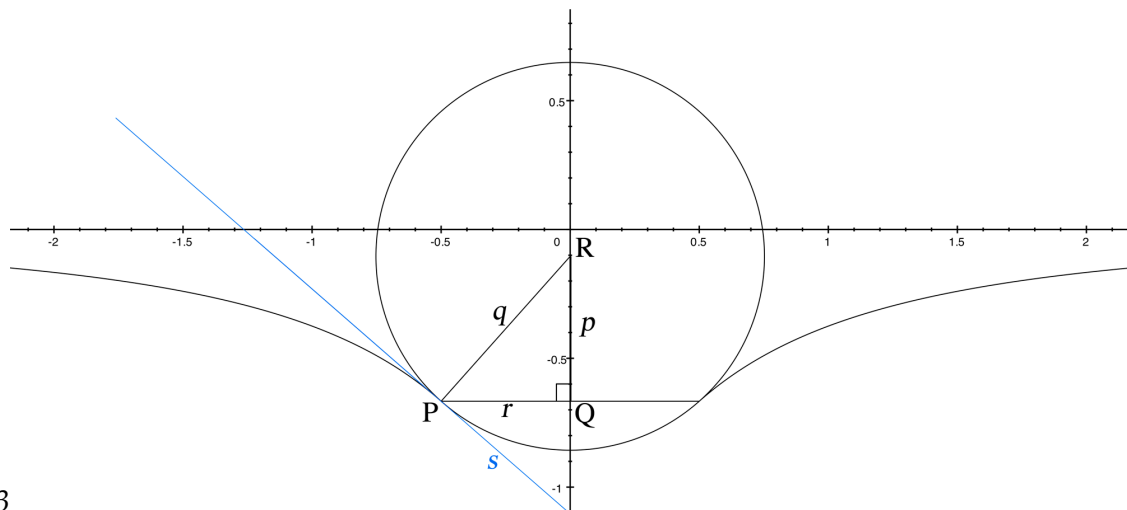


fig. 3

We know that value  $r$  is 0.5, which means point Q is at  $x=0$ . Because both P and Q lie on the horizontal line  $r$ , the  $y$  coordinate value of both P and Q can be found with  $-m(r+a)^{-2}$ , which is -0.666. To get the value of side  $p$  of the triangle, which will give us the needed  $y$  coordinate of point R, or circle center, we use

$$\frac{r}{\sin(\alpha R)} = \frac{q}{\sin(\alpha Q)} \Rightarrow \frac{0.5}{\sin(0.7266)} = \frac{q}{\sin\left(\frac{\pi}{2}\right)}$$

So  $\sin 0.7266 * q = r * \sin \pi/2$ . Therefore

$$q = \frac{r \left( \sin \frac{\pi}{2} \right)}{\sin(0.7266)}, \text{ which means } q=0.7526, \text{ which is the radius of circle needed to plot curve}$$

by equation 2. The length of  $p$ , using the Pythagorean theorem, is  $p = \sqrt{q^2 - r^2}$ . So the  $y$  coordinate value of R, the circle's center, is the  $y$  coordinate value of P +  $p$ , or  $-m(r+a)^2 + p$ , or -0.1042.

Now we can draw a circle with center R and radius  $q$ , and the slope at point P will be the tangent of the circle (fig. 4). This circle would describe the bowling ball as it rests on the rubber sheet. We have assumed a mass of  $m=1.5$ , but any value for  $m$  can be inserted to change the corresponding depth of the deformation of the rubber sheet. So now we have effectively combined the two equations, turning them on and off at the appropriate  $x$  values, to create the composite curve in figure 4.

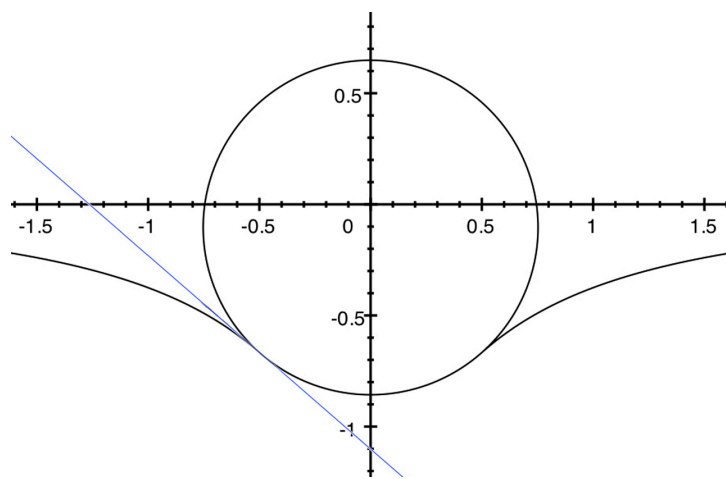


fig. 4

Equation 1 and 2 together, along with the binary switch to control them, looks like this in matrix form for graphing. Note  $t$  denotes the series variable that will be used to define the functions for  $x$  and  $y$ .

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1-1)t + 2q\cos u \\ (1-1)\left(-\frac{m}{(t+a)^2}\right) + 2\left(q\sin u + \left(-\frac{m}{(r+a)^2} + p\right)\right) \end{bmatrix}$$

where  $g=2m(|r+h|+a)^{-3}$ ,  $u=t\frac{3\pi}{2}-|\text{atang}|$ ,  $\alpha P=\text{atang}=\frac{\pi}{2}$ ,

$$\alpha Q=\frac{\pi}{2}, \alpha R=\pi-(\alpha P+\alpha Q), q=\frac{r(\sin\alpha Q)}{\sin\alpha R}, p=\sqrt{q^2-r^2}$$

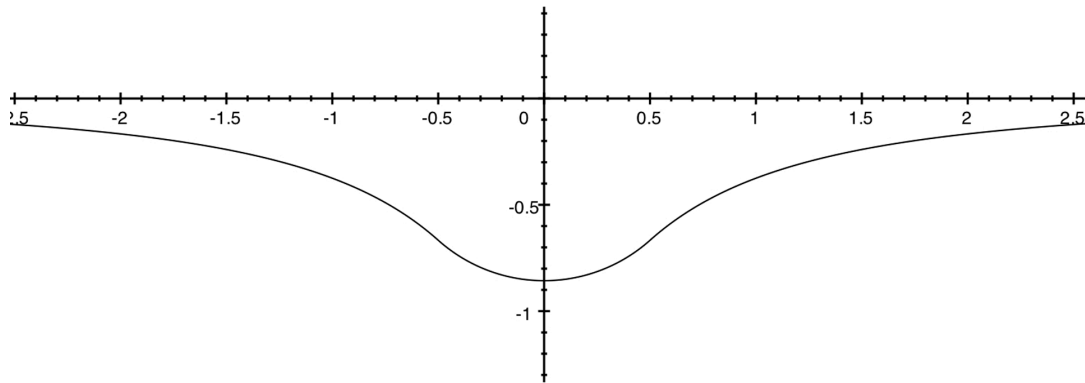
Note the use of  $(1-1)$  to turn on and off equation 1, and  $2$  to turn on and off equation 2.

Using the binary switch  $2$  to turn on the circle equation between  $t=-r$  and  $r$  requires that the value of  $u$  be a bit more complex, allowing all coordinates of a completed circle to be plotted, for any value of  $r$  chosen. Without the binary switch we could use a much simpler equation to draw the circle.

And with a further adjustment to the  $u$  value of equation 2, we can draw only the needed arc of the circle, to complete the composite curve without the rest of the circle above it. All that we need to do is change the value of  $u$  in a way which will then draw the arc of the circle only between  $x=-r$  and  $x=+r$ . Here is the necessary modification of the formula for  $u$  to achieve this.

$$u=\left(\frac{3\pi}{2}-|\text{atang}|\right)+(2|\text{atang}|\frac{t+r}{2r})$$

And the final composite curve, created by turning on and off the two equations with a binary switch, looks like this.



*fig. 5*

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