Throwing a Curve at Logarithmic Spirals

generating logarithmic spirals without using e

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April 30, 2016

Spirals are a natural form, existing in nature in all sizes, from microscopic shapes created by life to the structures of galaxies themselves. A while ago I became interested in logarithmic spirals. Also called equiangular spirals, these curves maintain the same angle at each point to the radii that cut across them. The great mathematician Jacob Bernoulli was so infatuated with logarithmic spirals that he had one engraved on his tombstone, though the engraver didn't get it right.

Logarithmic spirals are also called growth spirals, because the distances between each turn of the spiral increases geometrically. This makes a logarithmic spiral look the same regardless of the zoom level in or out from which it is viewed. This fractal-like self-similar quality, and its connection to nature, is what makes this type of spiral so fascinating.

The traditional way to generate a logarithmic spiral is by using e, the base of the natural logarithm. It is not surprising that the transcendental constant e is used for creating a growth spiral, because e is intimately involved in growth of all types, from natural life processes to the accruing of interest by your money. The standard equation to create a growth spiral is $r = ae^{b\theta}$, where a is a sizing constant and can be any number, e is the base of the natural logarithm, and b is the cotangent of the angle that will remain the same in relation to all radii vectors that cut across the spiral. θ is the angle of the radius for each point of the spiral in polar coordinates, not to be confused with the spiral's constant angle b. For this article I am going to explore equiangular curves with an equiangle of 45º, or $\pi/4$. Since the cotangent of $\pi/4$ is 1, we may dispense with the variable b in our equation. Removing the sizing variable a, we are left with $r = e\text{^}\theta$ to generate a logarithmic spiral with an constant angle of $\pi/4$.

Working with spirals, I became interested in creating a curve by dividing a radius into a fixed number of sections, defining squares along the radius. If we keep the contiguity of these discrete squares, but slide them the maximum distance apart from each other so that only the corners are touching, we can generate an equiangular curve that moves across these contiguous elements from corner to corner. In the example below I have divided a polar plane between radius 0.5 and 1 into 10, 60 and 600 elements respectively.

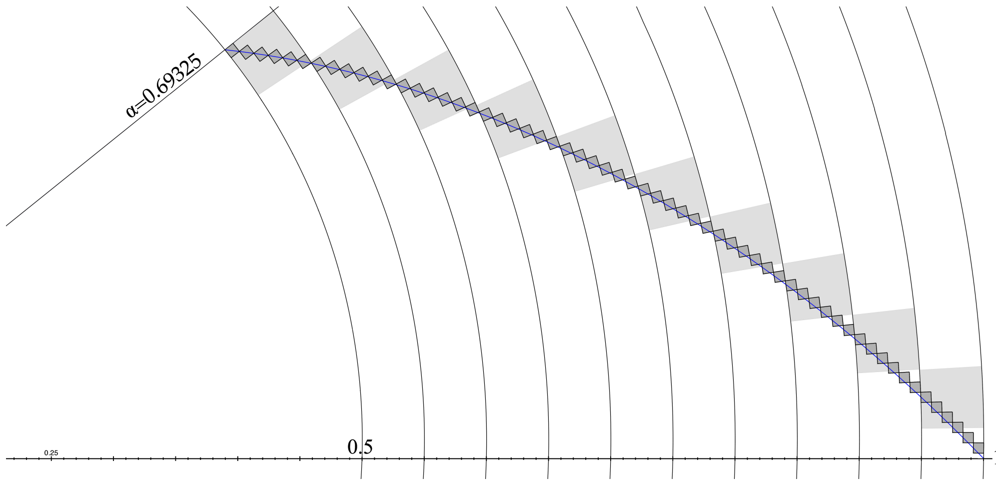

Figure 1

As we see, increasing the resolution, or number of squares, creates a curve that approaches an equiangular curve of 45º. I was interested to note that such an equiangular curve can be generated this way, without using the base of the natural logarithm. I discovered that using sigma summation to describe the resolution of the radial blocks could define the curve in figure 1 like this:

$$\theta = -\sum_{p=1}^{ru} \frac{1}{p}$$

where,
θ = angle in polar coordinates
r = radius
u = resolution of curve

To graph this equation, we can use the matrix form

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} t \\ -\sum_{p=1}^{tu} \frac{1}{p} \end{bmatrix}$$

where,
t = 0...10
u = 500

However, though we can create an equiangular curve with this equation, the starting angle of this curve varies with the resolution setting, or u. I wanted to have all resolution values of this curve cross $r = 1$ at $\theta = 0$. This problem was solved by adding an extra summation α that calculates the maximum value from summing all values of u, which, when β is subtracted from α, gives us the necessary starting angle regardless of the resolution chosen, to cross $r = 1$ at $\theta = 0$. Now we have

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} t \\ \alpha - \beta \end{bmatrix}, u = 120, d = 0.5, t = d...1, \beta = \sum_{p=1}^{tu} \frac{1}{p}, \alpha = \sum_{n=1}^{u} \frac{1}{n}$$

where I have added d to allow us to start the curve at any point along the radius. Here it starts at $r = 0.5$, as in figure 1. The curve generated is a segment of a logarithmic spiral.

Below in figure 2 we see the logarithmic spiral created on the left with the traditional equation, and on the right with my equation. It is clear that they are similar, and the spiral's growth has been described without the use of e, the base of the natural logarithm.

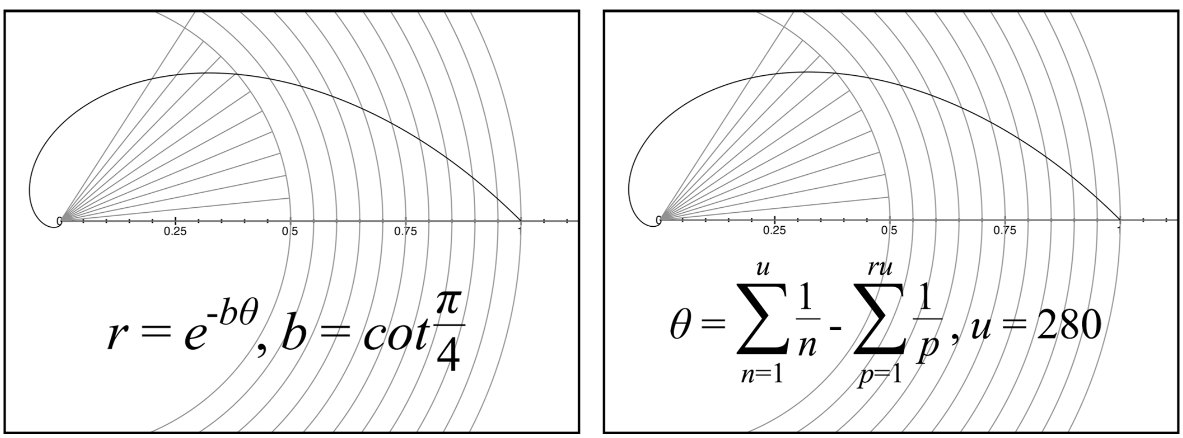

Figure 2

An interesting thing to note is that the spiral graphed by the tradition formula of r=e^-$b\theta$ plots only to r=1 and then stops. Changing the exponent from negative to positive will then graph beginning at r = 1 and continue from there. This is not surprising since e is the inverse of the natural logarithm, and ln(x) and ln(x^-1) return equivalent values but of opposite polarity. As we see in figure 3 below, this means that r = e^(-θ) will only return values between 0 and 1, and r = e^θ will only return values of >1. *(In figure 3 - variable b has been removed, which simply is equivalent to* b *= 1 or the cotangent of* π */ 4, the spiral form we are exploring).* The direction of the curve reverses direction when changing from r^-θ to r^θ, but this can be remedied by using r = e^(2π-$b\theta$).

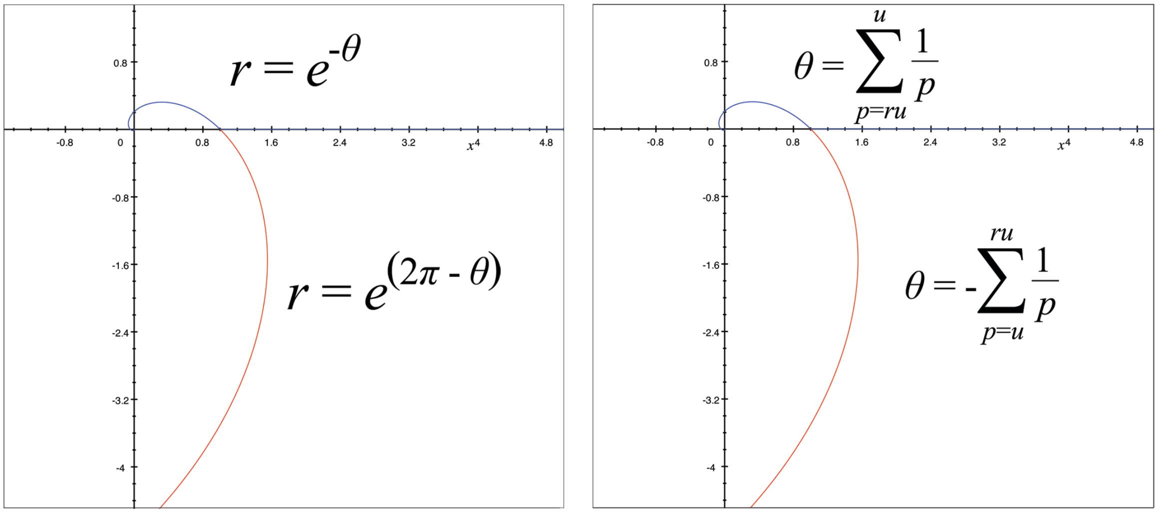

figure 3

My spiral equation can be reduced to a single summation $\theta = -\sum_{p=u}^{ru} \frac{1}{p}$, and we immediately see that now the equation only returns values of $r > 1$, the same as $r = e$^(2π-θ). The part of the spiral between $0 < r < 1$ can be calculated by modifying the single summation thus:

$$\theta = \sum_{p=ru}^{u} \frac{1}{p}$$

which is the same as $r = e^{-\theta}$ in figure 3.

What is of particular interest is that my double sigma formula on the right side of figure 2 calculates and plots the spiral for both $0 < r < 1$ and $r > 1$. This has implications that will be discussed in my next article.

To make a formula using a single summation that will create the entire spiral from $0 < r < 1$ and also from $r > 1$, we can make an equation that reacts differently for values of $0 < r < 1$ and for values $r > 1$, that gives the appropriate output:

$$\theta = -\frac{|ru - u|}{ru - u} \sum_{n=a}^{b} \frac{1}{n}$$

where,

$$a = \frac{(ru+u) - (|ru - u|)}{2}$$

$$b = a + (|ru - u|)$$

The switch $\frac{|ru-u|}{ru-u}$ is necessary to reverse the polarity of angle θ for values $r < 1$, else that segment of the spiral will be reversed from proper orientation. This switch serves the same purpose as changing the exponent in $r = e^\wedge\theta$ to $r = e^\wedge(2\pi\text{-}\theta)$. (figure 4)

$$\theta = -\frac{|ru - u|}{ru - u}\sum_{n=a}^{b}\frac{1}{n}, \quad a = \frac{(ru+u) - (|ru - u|)}{2}, \quad b = a + (|ru - u|)$$

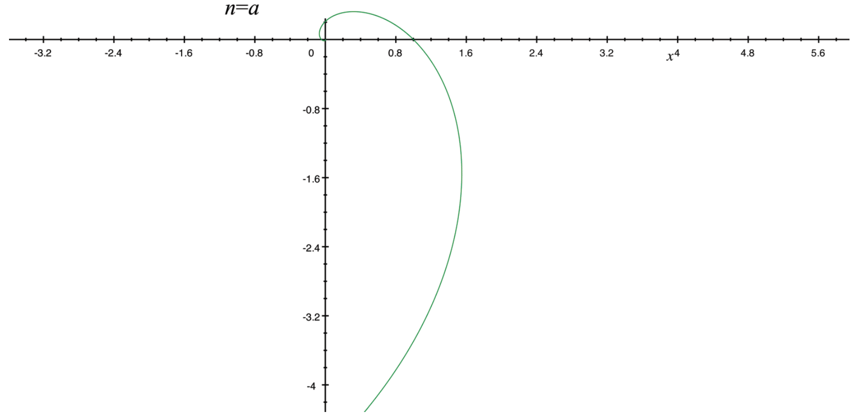

figure 4

Now we have created a logarithmic spiral with an equiangle of $\pi/4$ without using using e, and instead using sigma summation in its place. There were immediate implications from this summation equation concerning both e and the natural logarithm, and led to a discovery that I will explore in my next article.

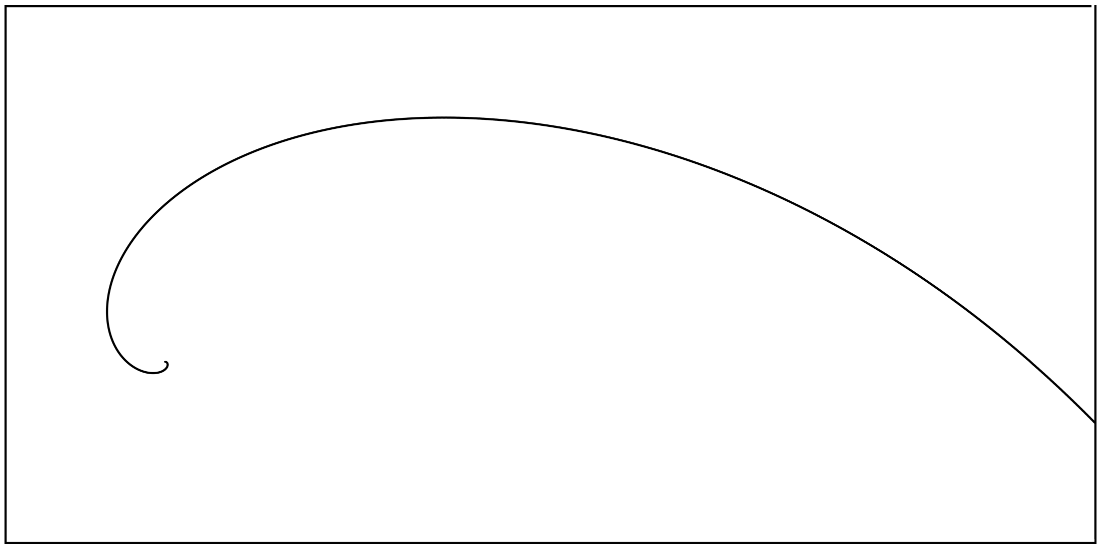

figure 5, log spiral graphed without *e*

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April 30, 2016, revised October7, 2016