Bowling Balls and Binary Switches

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Back in the old programming days, my favorite line of code was "if-then." If some condition is met, then do something specific - incredibly useful for making things happen exactly when you want them to happen. Well, recently I was using graphing software to generate curves and wanted to find an algebraic equivalent of an if-then command. Like a binary switch, it would return a value of 1 for a certain range of x-axis values, and return a value of zero for all other x-values. My desire was to be able to turn individual sub-equations on and off during the execution of a single graphing equation, while graphing a curve that has different functions at different places along the x-axis. Multiplying these different sub-equations by the binary switch equation would effectively turn them on and off wherever I wanted.

So I imagined a segment on the x-axis with a center at $x = 0$, and a variable r defining an equal length along both the positive and negative axis of x. When the x-value is between $-r$ and $+r$, I wanted an equation that would return a value of 1, and at all other x values would return a zero. An equation using absolute values does the trick - now I had a switch that could activate different equations at different places on the x-axis. Let's arbitrarily set the value of r to 0.5, giving the range between $x = -0.5$ and $x = +0.5$ that will be controlled separately by the binary switch. Let's call this binary switch ב, so

$$ב = \frac{(|x| - r) - |(|x| - r)|}{2(|x| - r)}$$

binary switch equation

As you can see, when $x \leq -r$ and $x \geq +r$, this binary switch formula will return a value of zero. When $x > -r$ and $x < +r$, it will return a value of one. A use I had in mind was plotting a simple curve with the equation:

$$-\frac{m}{(|x| + a)^2}$$

equation 1

which I will call *equation 1*. It gives the curve in fig. 1. I have chosen the arbitrary value of $m = 1.5$ to determine the depth of the curve, and the value $a = 1$ is necessary to keep all values of the denominator ≥ 1.

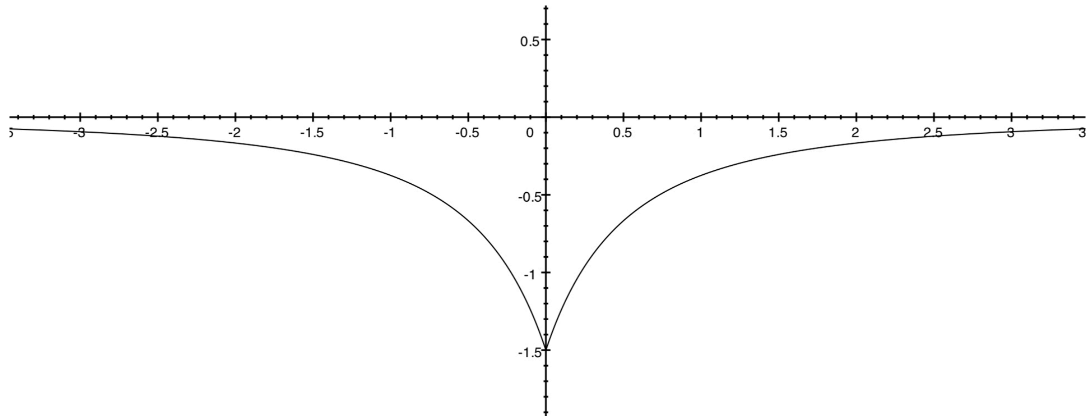

fig. 1: Initial Plot of Equation 1

What I'd like to do is have this curve be plotted until the value of $x = -r$ is reached, and then terminate, and then begin to plot again at $x = +r$, and continue. I can do this using the binary switch to turn *equation 1* on and off . However, since the binary switch ב returns a value of zero when $x > -r$ and $x < +r$, that would turn off *equation 1* when I need to turn it on. So I need to reverse the output of ב, changing 1 to zero and zero to 1. This I can do by simply subtracting 1 from ב and taking the absolute value, or $|\text{ב}-1|$. That way, when ב returns a zero, $|\text{ב}-1|$ returns 1, and when ב returns a 1, then $|\text{ב}-1|$ returns zero. Now, since

$$|\text{ב} - 1| = \left| \frac{(|x| - r) - |(|x| - r)|}{2(|x| - r)} - 1 \right|$$

I can use

$$(|\text{ב} - 1|)\left(-\frac{m}{(|x| + a)^2}\right)$$

equation 1b, with binary switch

to create the curve in figure 2 that is continuous when $x < -r$ *and* $x > r$, being undefined when $-r < x < r$.

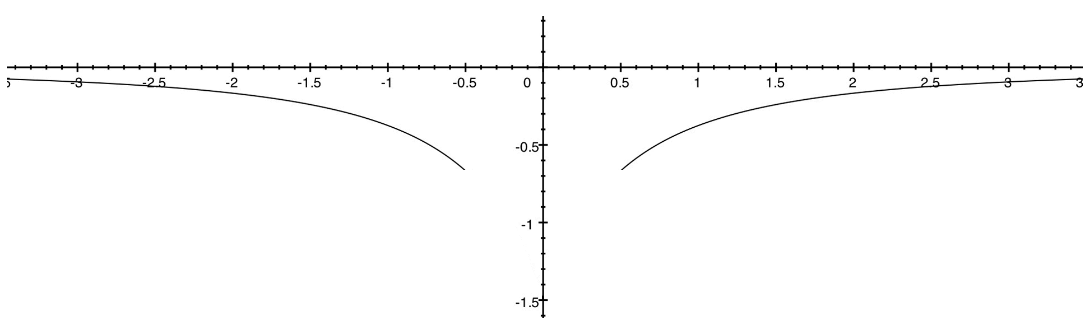

fig.2: Binary Switch function

Now a second equation is needed that will create a different curve in the missing spot along the x-axis, when $x > -r$ and $x < +r$. I imagined a rubber sheet with a bowling ball on it, with its center at $x = 0$. The curve down to the boundary of the ball has been created by equation 1, and the curve below the bowling ball will need a different equation, namely to create an arc of a circle cut by a chord at values $x = -r$ and $x = +r$. This curve, which will follow the bottom of the bowling ball, can be created by drawing a circle at a precise point, and then refining the equation to draw only the arc of the circle that connects $x = -r$ and $x = +r$. First let's look at a way to draw a circle in graphing software, in *equation 2* below. The value of q defines the size of the circle, and the value of u determines how much of the arc of the circle will be drawn. To draw the entire circle, u must be $\geq \pi$. The variable p will determine where on the y-axis the center of the circle will be.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q\cos(ut) \\ q\sin(ut) + p \end{bmatrix}, \text{ where } u \geq \pi \text{ and } t = -1 \ldots 1$$

equation 2

If $q = 0.5$, $p = 0$ and $u = \pi$, we see that we have a very nice circle with a radius of q and a center at $y = p$.

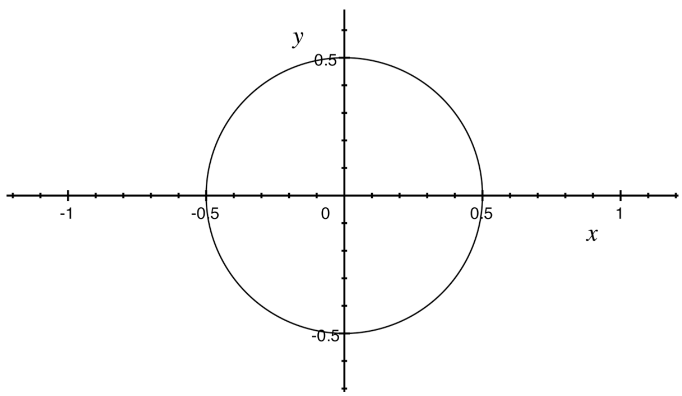

fig.2a: graph of simple circle with equation 2

Of course, this circle equation will have to be modified to find exactly where to draw the circle, and what size it will be; it will then have to be modified further to draw only the arc needed to fill in the curve in *figure 2*. The first step in this process is to find the location of the center of the circle to be drawn, or point R in *figure 3*.

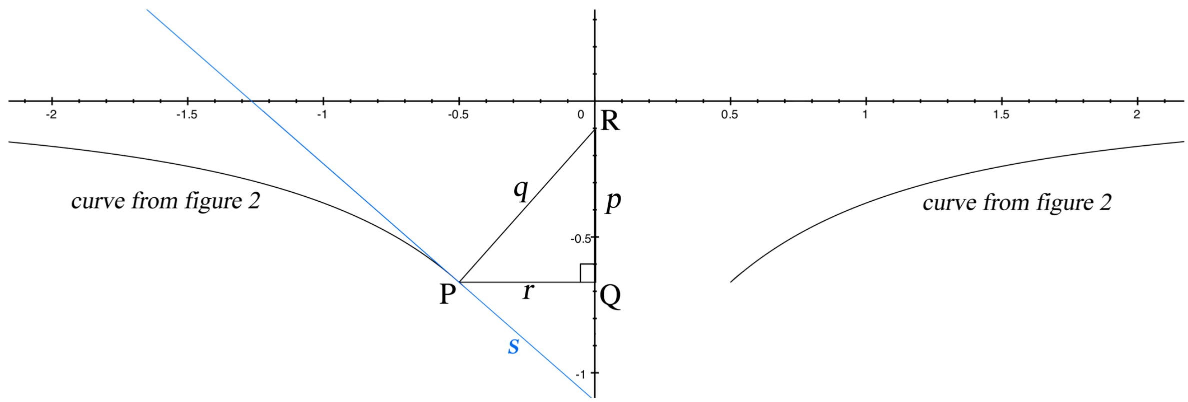

fig. 3: building the circle

Point P is the termination point of the curve in figure 2 along the negative x-axis. The y coordinate for point P (and point Q) is returned by $-m / (r + a)^2$, but we will need the length of line p to find the circle's center. To find p I began by calculating the first derivative of the termination point of *equation 1* at point P, or $x = -r$, to arrive at the instantaneous slope of the curve at point P. I do this because I will construct the circle such that the slope at point P will also be the slope of a tangent line to the needed circle. The radius q of that circle will be perpendicular to tangent line s and I will use the tangent to find the length of the radius of the circle. So, since we have chosen $r = 0.5$, then using the power rule for *equation 1*, we have:

$$y(t) = -\frac{m}{(|t|+a)^2}, \text{ where } m = 1.5,\ a = 1$$

$$-2m(|t| + a)^{-3}, \text{ or } \frac{dy}{dt}_{(t=-r)} = -0.888...$$

The measurement of the angle of the slope in relation to line r ($\angle$UPV in figure 4 below) is $\arctan\left(\frac{dy}{dt}\right)(t=-r)$, or $\arctan(-2m(|r|+a)^{-3})$, which equals -0.7266, or -41.63°.

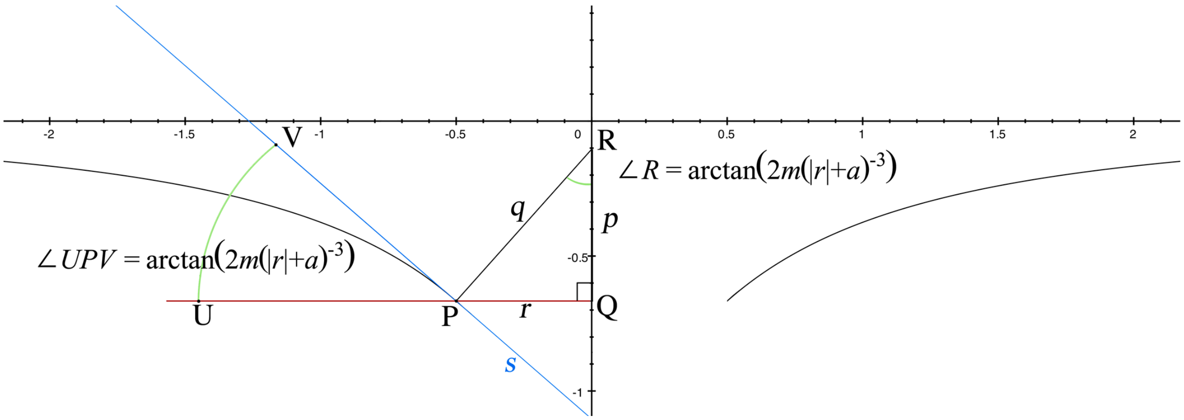

fig. 4: getting some angles

Since line q is at right angles to the slope or tangent line, adding the absolute values of $\angle sr$ and $\angle$P (figure 3) must equal $\pi/2$. Because $\arctan\left(\frac{dy}{dt}\right)(t = -r)$ is negative (since *equation 1* returns negative curvature on the y-axis), we simply add $\pi/2$, so $\angle P = \frac{\pi}{2} + \arctan\left(\frac{dy}{dt}\right)(t = -r)$, or $\angle$P = 0.8442, or 48.37º. Now, knowing that $\angle$Q is a right angle, we can find $\angle$R, which is π - ($\angle$P +$\angle$Q), or 0.7266. $\angle$R is also equivalent to $\arctan\left(\frac{dy}{dt}\right)(t = -r)$, so the absolute value of angles of $\angle$R and $\angle$UPV are identical (figure 3). Armed with this information, it is easy to use the trigonometric law of sines to get the x,y coordinates of point R, which is the center of the circle to be drawn.

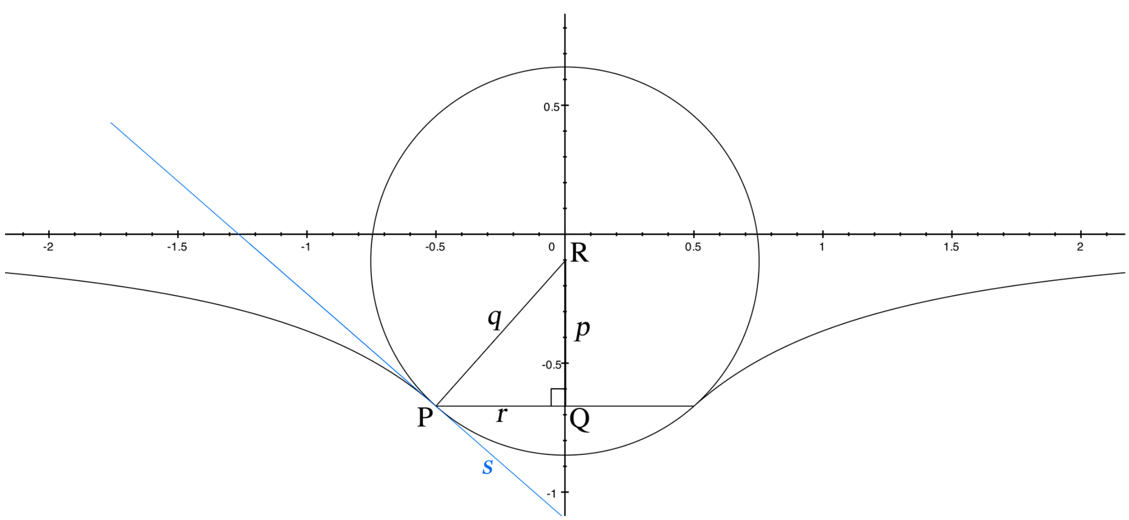

fig. 5: With the addition of the circle

We know that value r is 0.5, which means point Q is at $x = 0$. Because both P and Q lie on the horizontal line r, the y coordinate value of both P and Q can be found with $-m(r+a)^{-2}$, which is -0.666. To get the value of side p of the triangle, which will give us the needed y coordinate of point R, or circle center, we use

$$\frac{r}{\sin(\angle R)} = \frac{q}{\sin(\angle Q)} \Rightarrow \frac{0.5}{\sin(0.7266\ )} = \frac{q}{\sin\left(\frac{\pi}{2}\right)}$$

So (sin0.7266) * $q = r$ * sin(π/2)

therefore,

$$q = \frac{r\left(\sin\frac{\pi}{2}\right)}{\sin(0.7266)}$$

which means q=0.7526, which is the radius of circle needed to plot the curve by *equation 2*. The length of p, using the Pythagorean theorem, is $p = \sqrt{q^2 - r^2}$, or 0.5625. So the y coordinate value of R, the circle's center, is the y coordinate value of P plus the length of p, or $-m(r+a)^{-2} + p$, or -0.1042.

We can also find the circle's center by calculating the length of p using only ∠P, by comparing the length of adjacent side r of ΔRPQ to cos∠P, which is the adjacent/hypotenuse ratio of a classic triangle where hypotenuse = 1. The value of (cos∠P)/r gives us a comparative relation that, when acting as a divisor for sin∠P, returns the actual length of the opposite side of ΔRPQ, which is the length of p. So:

$$p = \sin\angle P \div \frac{\cos\angle P}{r}$$

A third solution for finding p is achieved by using the tangent of ∠R. Since we know ∠UPV and ∠R are equivalent, we know that tan∠UPV and tan∠R are also equivalent. And because $tan\angle R = \frac{opposite}{adjacent}$ sides of ΔRPQ, then tan∠R = r/p. Therefore:

$$p = \frac{r}{\tan\angle R}$$

which also returns a value for p of 0.5625.

By adding the length of p to the y coordinate value of Q, or $-m(r+a)^{-2}$, we arrive at the y coordinate value of R, which is again the circle's center. So:

$$m(r+a)^{-2} + \left(\sin\angle P \div \frac{\cos\angle P}{r}\right) = R_y$$

Now we want draw a circle with center R_y and radius q, and the slope at point P will be the tangent of the circle (fig. 6). This circle would describe the bowling ball as it rests on the rubber sheet. We have assumed a mass of m=1.5, but any value for m can be inserted to change the corresponding depth of the deformation of the rubber sheet. So now we are ready to combine and modify the two equations, turning them on and off at the appropriate x values, to create the composite curve in figure 6.

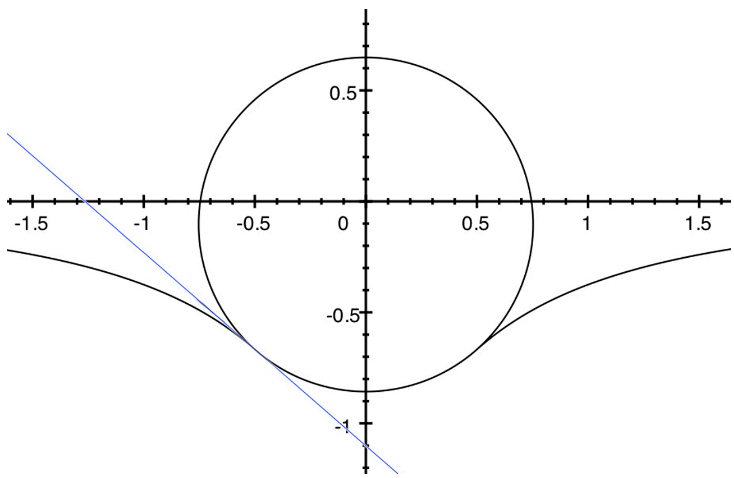

fig. 6

First we take *equation 2* and add *equation 1* into it, plus the value for p. By adding these two components to *equation 2*, we set the center of the circle at the proper coordinates to seamlessly connect to the termination at $-r$ and $+r$ of the curve drawn by *equation 1b* (which is *equation 1* turned on and off by the binary switch formula). We will call this composite formula *equation 3*.

$$q\sin(u) + \left(-\frac{m}{(r+a)^2}\right) + p$$

equation 3

To create the correct size of the circle in relation to the mass m and the value r, we have $u=\pi\ (t/r)$. It is important to remember that r does not represent the radius of the bowling ball, which is the value q (figure 5). Instead, $-r$ and r represent the points on the x axis at which the rubber sheet initiates contact with the bottom of the bowling ball. There is an interplay between the values m and r that affect the size of the bowling ball and the depth it sinks into the rubber sheet. For example, if r is made larger but the mass is not, the size of the bowling ball will increase substantially and the deformation it makes into the rubber sheet will decrease. This is understandable if we imagine the same mass being spread onto a larger area of the sheet, increasing elastic support due to greater surface area contact, hence reducing the deformation. Conversely, if we increase the mass or reduce the value for r and thus shrink the ball's circumference, it will sink deeper into the sheet.

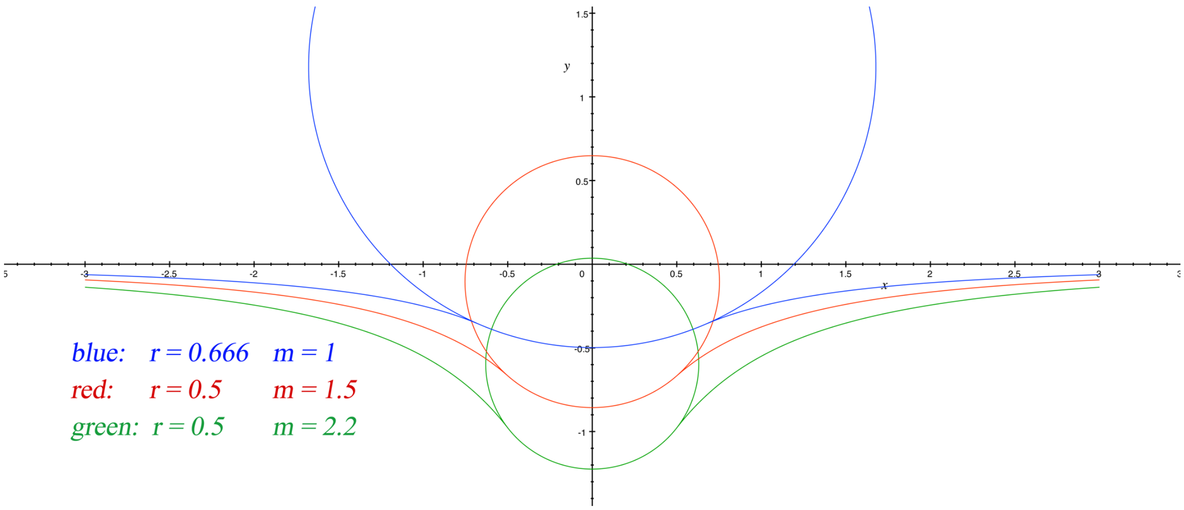

fig. 7

In figure 7 we see three examples of the effect of changing the values of m and r. To graph these curves, we first add the binary switch ב to *equation 3*, giving us *equation 3b:*

$$ב\left(q\sin(u)+\left(-\frac{m}{(r+a)^2}\right)+p\right)$$

equation 3b, with binary switch

Now we combine *equation 1b* and *equation 3b* together, which both have the binary switch ב to turn them on and off respectively at the x values of r and $-r$. Here is the final composite equation in matrix form for graphing. Note t denotes the series variable that will be used to define the functions for x and y.

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} (|ב-1|)t+בq\cos(u) \\ (|ב-1|)\left(-\frac{m}{(|t|+a)^2}\right)+ב\left(q\sin(u)+\left(-\frac{m}{(r+a)^2}+p\right)\right) \end{bmatrix}$$

where,

$$u = \pi \frac{t}{r}$$

equation 4, combining eq. 1b and eq. 3b

Note the use of $|ב\text{-}1|$ to turn on and off *equation 1*, and ב to turn on and off *equation 3*. Lest any eyebrows be raised at the momentary division by zero when $x = r$ and $x = -r$, I made this little formula to solve that issue. By the addition of $0^{|(|x|)-r|}$ to both the numerator and denominator of ב, when $x = r$ and $x = -r$ the binary switch will return a value of 1, since the accepted value of $0^0 = 1$, removing the specter of division by zero. Here also t denotes the series variable that will be used to define the functions for x and y.

$$ב = \frac{(|t| - r) - |(|t| - r)| + 0^{|(|t| - r)|}}{2(|t| - r) + 0^{|(|t| - r)|}}$$

binary switch with undefined value at r and -r resolved

Now that we can draw the full bowling ball circle, we want to remove the ball and draw only the curve of the rubber sheet where it makes contact with the ball. To draw only the necessary arc of the circle to describe the bowling ball's contact with the rubber sheet, the value for u has to become a bit more complex. We begin with arctan($-2m(|r|+a)^{-3}$) again for ∠UPV and multiply it by t/r. This will give us the proper arc length we need, but in the wrong rotation along the circumference of the circle. So we need to include $(3/2)\,\pi$ to place the curve in the proper orientation to connect to the termination points of the curve from *equation 1*. Predictably, each increment of $(1/2)\,\pi$ rotates the curve segment 90º around the circle, with 2π making a full rotation. $(1/2)\,\pi$ puts the curve at top, π puts curve on left side, and $(3/2)\,\pi$ puts it at bottom where we want it. So we have:

$$u = \frac{3}{2}\pi - \frac{t}{r}\left|\operatorname{atan}\left(-2m(r + a)^{-3}\right)\right|$$

With the necessary value of u in hand, we have our final composite equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (|ב - 1|)t + בq\cos(u) \\ (|ב - 1|)\left(-\frac{m}{(|t| + a)^2}\right) + ב\left(q\sin(u) + \left(-\frac{m}{(r + a)^2} + p\right)\right) \end{bmatrix}$$

where,

$$u = \frac{3}{2}\pi - \frac{t}{r}\left|\operatorname{atan}\left(-2m(r + a)^{-3}\right)\right|$$

$$\angle P = \frac{\pi}{2} + \operatorname{atan}\left(-2m(r+a)^{-3}\right)$$

$$\angle Q = \frac{\pi}{2}$$

$$\angle R = \pi - (\angle P + \angle Q)$$

$$q = \frac{r(\sin\angle Q)}{\sin\angle R}$$

$$p = \sqrt{q^2 - r^2}$$

equation 4b, drawing final curve

And the final composite curve, created by turning on and off the elements of the composite equation with a binary switch, looks like this.

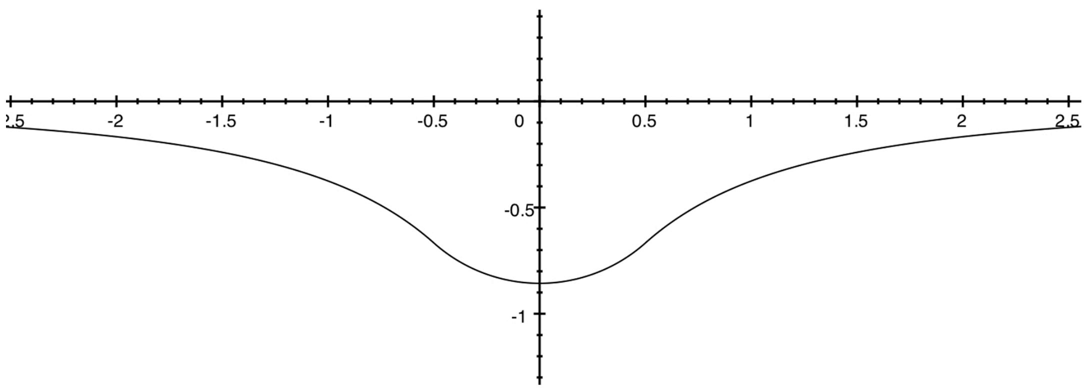

fig. 8: Final Figure

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